UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
Department of Electrical & Computer Engineering

ECE 204 *Numerical methods*

The bracketed secant method
a.k.a. the false-position method or *regula falsi*

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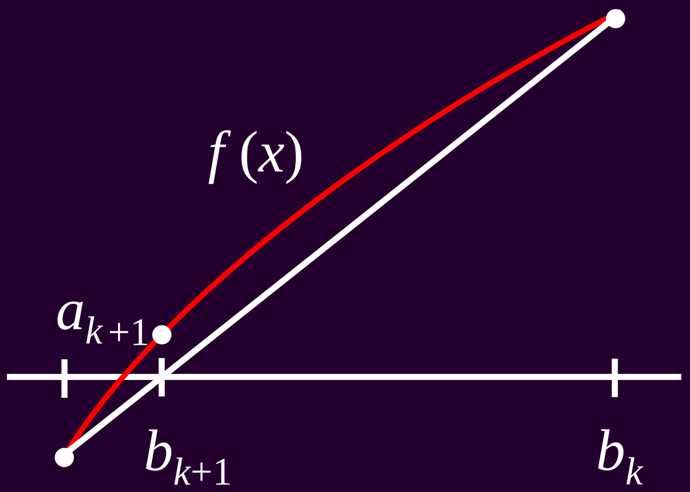
Introduction

- In this topic, we will
  - Describe a modification to the bisection method
    - Find the root of the interpolating linear polynomial
  - Try to determine the rate of convergence with an empirical example
  - Discuss some differences
  - Look at an implementation

A secant line

- Suppose we have a real-valued function of a real variable and suppose that a_k and b_k are two points such that $f(a_k)$ and $f(b_k)$ have opposite signs
  - If f is continuous, then f must have a root on $[a_k, b_k]$



A secant line

- What is the root of the line connecting $(a_k, f(a_k))$ and $(b_k, f(b_k))$?

$$f(a_k) + \frac{f(b_k) - f(a_k)}{b_k - a_k}(x - a_k)$$

$$f(a_k) + \frac{f(b_k) - f(a_k)}{b_k - a_k}(x - a_k) = 0$$

$$\frac{f(b_k) - f(a_k)}{b_k - a_k}(x - a_k) = -f(a_k)$$

$$x - a_k = -f(a_k)\frac{b_k - a_k}{f(b_k) - f(a_k)}$$

$$x = a_k - f(a_k)\frac{b_k - a_k}{f(b_k) - f(a_k)}$$

A secant line

- What is the root of the line connecting $(a_k, f(a_k))$ and $(b_k, f(b_k))$?
  - This formula is usually simplified to

$$r_k \leftarrow \frac{f(b_k)a_k - f(a_k)b_k}{f(b_k) - f(a_k)}$$

  - This, however, is subject to subtractive cancellation
- Thus, we will adopt

$$r_k \leftarrow a_k - f(a_k)\frac{b_k - a_k}{f(b_k) - f(a_k)}$$

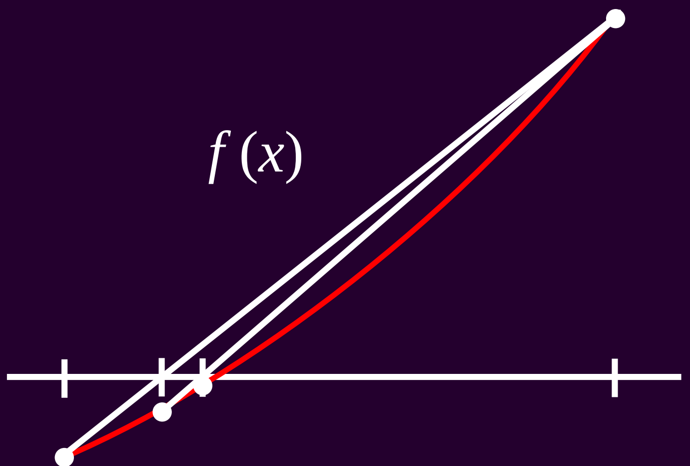
$$r_k \leftarrow a_k - f(a_k)/f^{(1)}(a_k)$$

A secant line

- Thus, given two bounds a_k and b_k such that $f(a_k)$ and $f(b_k)$ have opposite signs
  - Find the root
    $$r_k \leftarrow a_k - f(a_k)\frac{b_k - a_k}{f(b_k) - f(a_k)}$$
  - If $f(r_k) = 0$, we have found a root, so we are done
  - If $f(a_k)$ and $f(r_k)$ have opposite signs,
    let $a_{k+1} \leftarrow a_k$ and $b_{k+1} \leftarrow r_k$
  - Otherwise, $f(r_k)$ and $f(b_k)$ have opposite signs,
    let $a_{k+1} \leftarrow r_k$ and $b_{k+1} \leftarrow b_k$

A secant line

- How does convergence occur?
  - One of the end-points usually becomes fixed
    - Thus, in this example $b_k - a_k > b_k - r$
  - In theory, the root could be arbitrarily close to the other end-point, so the maximum error could be $b_k - a_k$
  - However, we will keep iterating until $r_k - a_k < \varepsilon_{\text{step}}$ or $b_k - r_k < \varepsilon_{\text{step}}$



Example

- Find the first root of $2e^{-2x} - e^{-x}$
  - The solution is ln(2)

k	a_k	b_k	$f(a_k)$	$f(b_k)$	r_k	$f(r_k)$	$\ln(2) - r_k$	$\frac{\ln(2) - r_k}{\ln(2) - r_{k-1}}$
0	0	1	1	–0.09721	0.9114034921336616	–0.07882	–0.2183	
1	0	0.911403492133662	1	–0.07882	0.8448178934459362	–0.06046	–0.1517	0.6949
2	0	0.844817893445936	1	–0.06046	0.7966507111390642	–0.04433	–0.1035	0.6824
3	0	0.796650711139064	1	–0.04433	0.7628346587707037	–0.03139	–0.06969	0.6733
4	0	0.762834658770704	1	–0.03139	0.7396168052064190	–0.02167	–0.04670	0.6668
5	0	0.739616805206419	1	–0.02167	0.7239275935246550	–0.01470	–0.03078	0.6624
6	0	0.723927593524655	1	–0.01470	0.7134425805685035	–0.009844	–0.02030	0.6594
7	0	0.713442580568504	1	–0.009844	0.7064881958397252	–0.006539	–0.01334	0.6573
8	0	0.706488195839725	1	–0.006538	0.7018989029405253	–0.004319	–0.008752	0.6560
9	0	0.701898902940525	1	–0.004319	0.6988805733976142	–0.002842	–0.005733	0.6551
10	0	0.698880573397614	1	–0.002842				

Error analysis

- It seems the error is dropping by a factor of 0.65
  - Thus is a constant times the previous error, so O(h)
  - The issue is that one end-point is fixed
  - Solution?
    - Alternate between the bracketed secant method and the bisection method

Implementation

```
double bracketed_secant( double f( double x ), double a, double b,
                         double eps_step, double eps_abs,
                         unsigned int max_iterations ) {
    assert( a < b );

    double fa{ f( a ) };
    double fb{ f( b ) };

    if ( !std::isfinite( fa ) || !std::isfinite( fb ) ) {
        return NAN;
    }

    if ( fa == 0.0 ) {
        return a;
    }

    if ( fb == 0.0 ) {
        return b;
    }
```

C/C++ Code is provided to demonstrate the straight-forward nature of these algorithms and not required for the examination

Implementation

```
for ( unsigned int k{0}; k < max_iterations; ++k ) {
    double r{ a - fa*(b - a)/(fb - fa) }; // This avoids overflow
    double fr{ f( r ) };

    if ( !std::isfinite( fr ) ) {
        return NAN;
    }

    if ( fr == 0.0 ) {
        return r;
    } else if ( std::signbit( fa ) == std::signbit( fr ) ) {
        if ( ((r - a) < eps_step) && (std::abs( fr ) < eps_abs) ) {
            return r;
        }

         a =  r;
        fa = fr;
    } else {
```

$r_k - a_k < \varepsilon_{\text{step}}$ and $\left|f\left(r_k\right)\right| < \varepsilon_{\text{abs}}$, return r_k

C/C++ Code is provided to demonstrate the straight-forward nature of these algorithms and not required for the examination

Implementation

```
        } else {
            if ( ((b - r) < eps_step) && (std::abs( fr ) < eps_abs) ) {
                return r;
            }

             b =  r;
            fb = fr;
        }
    }

    return NAN;
}
```

$b_k - r_k < \varepsilon_{\text{step}}$ and $\left|f\left(r_k\right)\right| < \varepsilon_{\text{abs}}$, return r_k

Summary

- Following this topic, you now
  - Considered a modification to the bisection method
    - Find the root of the interpolating linear polynomial
  - Looked at an example
  - Understood that the rate of convergence is no better than the bisection method
  - Considered an implementation

References

[1] https://en.wikipedia.org/wiki/Regula_falsi

Acknowledgments

Tazik Shahjahan for pointing out typos.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

https://www.rbg.ca/

for more information.

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